Heat Conduction in Elliptical Cylinders and Cylindrical Shells

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Solutions are obtained, by means of the Galerkin method, used together with the Laplace transform, for the transient temperature distribution in an elliptical domain and a cylindrical shell. A one-term and a three-term approximation are employed for the ellipse and a one-term approximation for the cylindrical shell. The problem of heating of a shell casing caused by the burning of a solid fuel is considered for the cases of both nonsymmetrical radial burning and end burning. Results are given for the symmetrical case to compare with the available published finite-difference results. The comparison is quite favorable.

Nomenclature

x, y = Cartesian coordinates (elliptical cylinder)
r, ψ, z = cylindrical coordinates (cylindrical shell)
t = time
$\theta(x, y, t), \theta(r, \psi, z, t)$ = temperature distribution
$\bar{\theta}(x, y; s), \bar{\theta}(x, y, z; s)$ = Laplace transform of θ
θ_0 = initial constant temperature for elliptical cylinder; exterior ambient temperature for cylindrical shell
θ_b = burning temperature of solid fuel
θ_{avg} = $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f_2(\psi)d\psi$
θ_g = temperature near inside surface of cylindrical casing
ψ = temperature function defined by $\bar{\theta} - (\theta_0/s)$
d, b = major and minor axes of elliptical cylinder
ρ = radius of hole in elliptical cylinder
r_i, r_e = interior and exterior radii of cylindrical shell
α = diffusivity
K = conductivity
h_1, h_2 = convective heat transfer coefficient
U = linear burning velocity of solid fuel
T = nondimensional temperature distribution; for elliptical cylinder $T = 1 - (\theta/\theta_0)$, for cylindrical shell with radial burning $T = \theta/\theta_{\text{avg}}$, for cylindrical shell with axial burning $T = \theta/\theta_b$
τ = $\alpha t/r_i^2$ = nondimensional time
R = r/r_i = nondimensional radius
Z = z/r_i = nondimensional axial coordinate
H = $h_1 r_i/K$ = nondimensional convective heat transfer coefficient
V = $Ur_i/\alpha L$ = nondimensional burning velocity of solid fuel
L = linear operators defined in Eqs. (6, 35, and 52)
$A_{nm}, b_p, b, c, \alpha_1, \alpha_2$ = coefficients in approximate solution
A_n, a_m, B_m = Fourier coefficients
ζ = Ut = location of burning surface at time t
$f_2(\psi)$ = arbitrary function of ψ (cylindrical shell)
$f_1(s)$ = Laplace transform of $f_1(t)$
$g(r, \psi)$ = arbitrary initial temperature distribution in shell
$\zeta(R; s), \eta(\psi; s)$ = functions defined in Eq. (29)
$\Gamma(R; s), \omega(Z; s)$ = functions defined in Eq. (49)
$\sigma(R)$ = generalized temperature distribution
λ = function satisfying given boundary conditions
ϕ = function satisfying homogeneous boundary conditions
β = coefficient defined in Eq. (60)
$P_k(R)$ = polynomials in R
$\bar{C}_k$ = coefficients, multipliers of $P_k(R)$
Q, u, v = coefficients defined in Eq. (66)

Received by ARS August 8, 1962; revision received February 14, 1963. This paper is a result of research sponsored by the National Science Foundation under NSF Grant 5948.

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Introduction

In Ref. 1 the authors showed how the Galerkin technique, when used in conjunction with the Laplace transform, may be applied to problems in heat conduction for a wide class of geometrical shapes for which the heat equation cannot be separated. It frequently happens, however, that, even when the equation of transient heat conduction can be separated, the resulting ordinary differential equations yield nonelementary functions, the properties of which have not been studied or tabulated. When these nonelementary functions involve one or more parameters, even this solution becomes essentially numerical in nature, requiring the computation of a new table for each different value of the parameter. In Ref. 2, for example, in studying the transient heat conduction in elliptical plates and cylinders, Kirkpatrick et al., after applying the method of separation of variables and obtaining two ordinary differential equations, were confronted with Mathieu functions for which there were no published values. It was necessary, therefore, to compute numerically the solution function using a high-speed digital computer; the solution so obtained is, of course, valid only for the particular values of the parameter (the eccentricity) considered. Similarly, in Ref. 3 Hatch et al. used a finite difference scheme to obtain temperature distributions caused by radial heating of a cylindrical shell. In this case the classical solution would yield Bessel functions.

In this paper it is shown how solutions may be obtained for the transient temperature distribution in elliptical cylinders and cylindrical shells with little effort by using the Laplace-Galerkin scheme. Two problems are considered, related to those studied in Refs. 2 and 3. By using this method it is possible to generalize the problems, and this has been demonstrated by including 1) an arbitrarily located circular hole in an elliptical cylinder and 2) angular and axial temperature variations in the cylindrical shell.

Heat Conduction in Elliptical Cylinders

The problems considered in the following two sections are concerned with the quenching phenomenon. That is, a body initially at some temperature is thrust quickly into a bath at some different temperature, and it is required to find the temperature distribution at any time. It is assumed that by means of continuous circulation or convection a very high boundary conductance is maintained so that the temperature at the boundary is equal to the bath temperature at all times after the initial instant.

The first problem considered is that solved by Kirkpatrick. The results obtained are compared to the latter's tabular values in Ref. 2. A second problem considered includes an arbitrarily located circular hole in an elliptic plate. There are no published values for this problem since, by its nature, the method of separation of variables cannot be applied in the first place. In each case it is assumed that the faces of the plate or cylinder are insulated so that there is no temperature variation through its thickness.

Quenching of a Solid Ellipse

Consider the two dimensional elliptical domain shown in Fig. 1, together with the following:

$$(\partial^2\theta/\partial x^2) + (\partial^2\theta/\partial y^2) = (1/\alpha)(\partial\theta/\partial t) \qquad t > 0 \quad (1)$$

$$\left.\begin{aligned}\theta(P, t) &= \theta_0\\ \theta(x, y, 0) &= 0\end{aligned}\right. \quad P \text{ on boundary} \quad (2)$$

This is the problem studied by Kirkpatrick. Applying the Laplace transform to eliminate the time dependence yields

$$(\partial^2\bar\theta/\partial x^2) + (\partial^2\bar\theta/\partial y^2) = (s\bar\theta/\alpha) \quad (3)$$

$$\bar\theta(P, s) = \theta_0/s \qquad P \text{ on boundary} \quad (4)$$

where

$$\bar\theta = \int_0^\infty e^{-st}\,\theta(x, y, t)dt$$

To make the boundary conditions homogeneous, assume a solution of the form

$$\bar\theta = \psi(x, y; s) + (\theta_0/s)$$

where from (4)

$$\psi(P; s) = 0 \quad (5)$$

Substituting for $\bar\theta$ in (3) yields

$$L(\psi) \equiv (\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) - (s/\alpha)\psi = \theta_0/\alpha \quad (6)$$

To apply the Galerkin method, a form of solution first is assumed which satisfies (5), and this is accomplished readily by the following:

$$\psi = [(x^2/d^2) + (y^2/b^2) - 1]\sum^\infty a_p\phi_p(x, y) \quad (7)$$

where the ϕ_p's constitute the elements of any complete set.

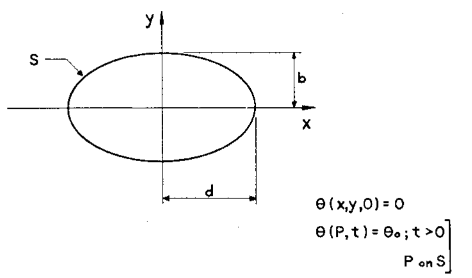


Fig. 1 Quenching of solid ellipse

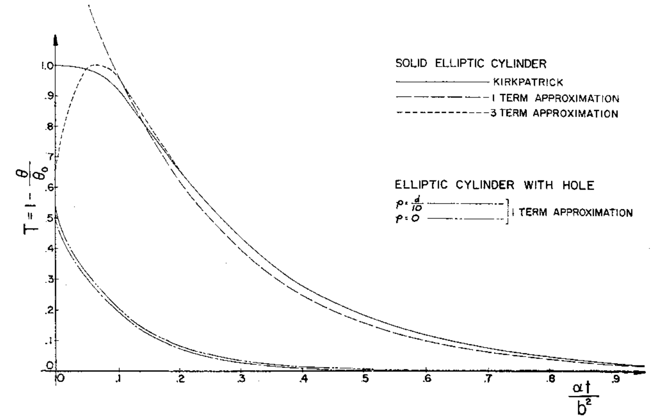


Fig. 2 Temperature difference at center of ellipse

The set chosen herein were the powers of x and y, so that rewriting (7)

$$\psi = \left(\frac{x^2}{d^2} + \frac{y^2}{b^2} - 1\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\,x^ny^m$$

Since it is not possible, in general, to consider the entire doubly infinite set, the series is truncated, and the approximate solution is then

$$\psi^{(rs)} = \left(\frac{x^2}{d^2} + \frac{y^2}{b^2} - 1\right)\sum_{n=0}^{r}\sum_{m=0}^{s} A_{nm}\,x^ny^m \quad (8)$$

The coefficients then are evaluated by the Galerkin algorithm, by substituting (8) into (6), multiplying by each ϕ in turn, and equating the resulting scalar products to zero, i.e.,

$$\iint_A\left[L(\psi^{(rs)}) - \frac{\phi_0}{\alpha}\right]\phi_p dxdy = 0$$

This yields a number of algebraic equations equal to the number of ϕ_p's originally included in (8) so that the A_{nm} are determined once the solution to the simultaneous system is known.

Using the foregoing procedure the one-term approximation yields‡

$$A_{00}{}^{(1)} = \frac{3}{2}\,\frac{\theta_0}{\{3\alpha[(1/d^2) + (1/b^2)] + s\}}$$

so that, after inverting, the solution is

$$\theta^{(1)} = \theta_0 + \tfrac{3}{2}\theta_0[(x^2/d^2) + (y^2/b^2) - 1]e^{-3\alpha[(1/d^2)+(1/b^2)]t} \quad (9)$$

Defining, as does Kirkpatrick, the temperature difference $T \equiv 1 - (\theta/\theta_0)$, it is seen that

$$T^{(1)} = -\tfrac{3}{2}[(x^2/d^2) + (y^2/b^2) - 1]e^{-3\alpha[(1/d^2)+(1/b^2)]t} \quad (9')$$

for the one-term approximation. This is plotted in Fig. 2 for an eccentricity of 0.7, i.e., $b^2 = 0.51d^2$, and dimensionless times $\alpha t/b^2$ between 0.1 and 1.0.

To obtain higher approximations, additional terms are added to the approximate solution (8). Because of symmetry A_{01}, A_{10}, and A_{11} must be zero. Adding the A_{20} and A_{02} terms gives as the approximate solution

$$T^{(2)} = [(x^2/d^2) + (y^2/b^2) - 1] \times [A_{00}{}^{(2)} + A_{20}{}^{(2)}(x/d)^2 + A_{02}{}^{(2)}(y/d)^2] \quad (10)$$

where the length scale has been made dimensionless by division by the length of the semimajor axis. Performing

‡ The superscript in parentheses indicates the approximation; $A_{00}{}^{(2)}$ for example, is the first coefficient of the second approximation.

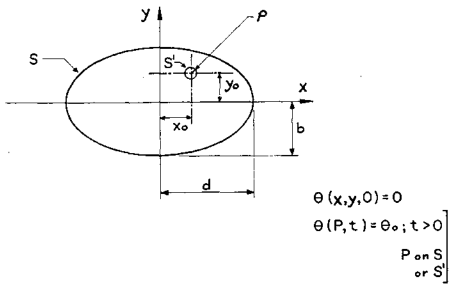


Fig. 3 Quenching of ellipse with arbitrarily located circular hole

the analogous operations as was done for the one-term approximation yields for the coefficients (eccentricity = 0.7)

$$A_{00}^{(2)} = 1.5956e^{-4.3539(\alpha t/b^2)} - 0.2808e^{-19.0507(\alpha t/b^2)} - 0.6481e^{-31.4587(\alpha t/b^2)}$$

$$A_{20}^{(2)} = -0.8451e^{-4.3539(\alpha t/b^2)} + 2.9597e^{-19.0507(\alpha t/b^2)} + 1.2187e^{-31.4587(\alpha t/b^2)}$$

$$A_{02}^{(2)} = -0.9597e^{-4.3539(\alpha t/b^2)} - 0.8881e^{-19.0507(\alpha t/b^2)} + 8.3810e^{-31.4587(\alpha t/b^2)}$$

This three-term solution also is plotted in Fig. 2, and it is seen that, whereas the one-term solution already gives results that would be satisfactory for most engineering applications, the three-term solution is hardly distinguishable from the Kirkpatrick result for dimensionless times $\alpha t/b^2$ greater than about 0.2.

Quenching of Ellipse with Hole

The statement of this boundary value problem (Fig. 3) is again as given by Eqs. (1–6); P here, however, refers to points on both the external and internal boundaries. In this case, the front factor of (7) must be modified so that ψ attains zero value on the internal boundary as well as the external boundary. This is accomplished easily.§ Assuming that the hole is circular, for example, of radius ρ, the center of which is at (x_0, y_0), the counterpart of (7) for this problem is

$$\psi = [(x - x_0)^2 + (y - y_0)^2 - \rho^2] \times [(x^2/d^2) + (y^2/b^2) - 1] \; \Sigma b_p \phi_p(x, y) \quad (11)$$

For a one-term approximation, the summation in (11) is replaced by $B_{00}^{(1)}$. For an ellipse of eccentricity equal to 0.7 and for arbitrarily selected values of $x_0 = 3d/10$, $y_0 = 3d/10$, and $\rho = d/10$, the solution is

$$T^{(1)} = 2.8623e^{-9.3228(\alpha t/b^2)} [1 - (x^2/d^2) - (y^2/b^2)] \times \{[(x/d) - 0.3]^2 + [(y/d) - 0.3]^2 - 0.01\} \quad (12)$$

This is plotted for the point (0, 0) in Fig. 2, from which one easily may observe the increased rate of cooling (or heating) caused by the presence of the hole located at $(0.3d, 0.3d)$. A similar plot for the point $(-0.57d, -0.155d)$ is shown in Fig. 4. This latter point is the approximate location at which (12) yields its maximum value.

The influence of the diameter of the hole may be determined by allowing the radius to shrink to zero, thus producing a line heat source at (x_0, y_0) in a solid ellipse. Setting ρ equal to zero gives as the solution function for T

$$T^{(1)} = 2.8361e^{-9.1960(\alpha t/b^2)} [1 - (x^2/d^2) - (y^2/b^2)] \times \{[(x/d) - 0.3]^2 + [(y/d) - 0.3]^2\} \quad (13)$$

This is plotted, as a function of time, for (0, 0) and $(-0.57d, -0.155d)$ in Figs. 2 and 4, respectively. As may be seen at both locations, the line source is virtually as effective as the finite hole in reducing the transient temperature difference

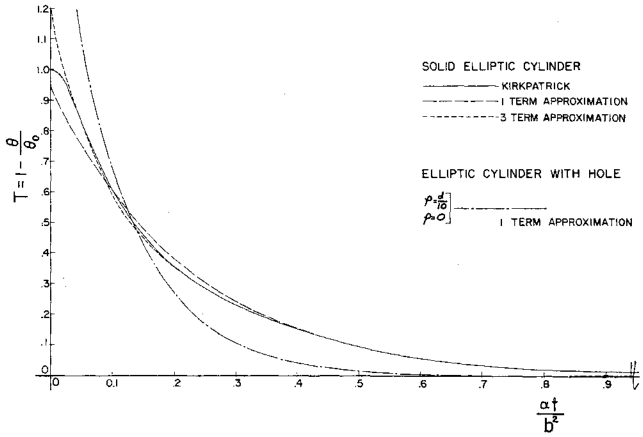


Fig. 4 Temperature difference at $(-0.57d, \; -0.155d)$

T. It is noted in passing that the solution obtained by assuming a form of (13) plus (9′) is an approximation to Green's function for an elliptical domain associated with the heat equation.

Heat Conduction in Cylindrical Shells

The second problem considered involves the heat conduction in a cylindrical shell caused, for example, by the burning of a solid fuel in a rocket engine.

There are two methods in which the fuel may be burned. In the first and more conventional manner, the fuel, cylindrical in shape and containing an axial hole, is fitted snuggly inside the shell (Fig. 5). The burning proceeds in a radial direction, although, owing to the noncircular shape of the hole, the temperature impressed on the inside of the casing has angular dependence. Near the inside surface of the casing, therefore, the temperature distribution is of the form

$$\theta_g = f_1(t) f_2(\psi) \quad (14)$$

where ψ is as shown in Fig. 6.

In the second method, the cylinder of fuel is solid, and the burning occurs across an entire face, proceeding from one end to the other in cigarette fashion (Fig. 7). Solutions

Fig. 5 Radial burning

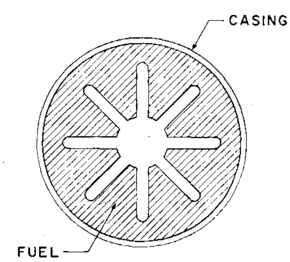


Fig. 6 Radial burning

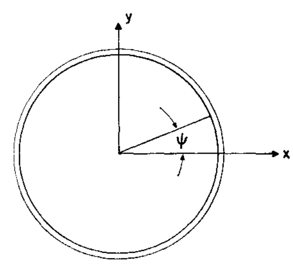
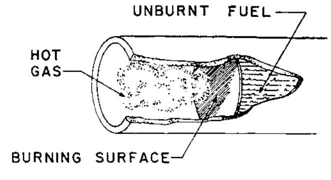


Fig. 7 End burning



§ See, however, the conclusion for a discussion of the restrictions on the shape of the interior boundary.

for both of these problems may be obtained formally in terms of Bessel functions. To use these solutions, however, would require extensive tables of Bessel functions for all the geometries encountered in practice; therefore, the formal solution must yield to approximate workable solutions. In Ref. 3, for example, solutions to the problem of radial heating of cylindrical shells (with no angular dependence) have been obtained numerically for a number of wall thicknesses and physical constants. The problem solved, however, is essentially a one-dimensional one; to include angular or axial variations would necessitate extensive numerical computation and curve plotting for each curve in Ref. 3.

Using the method previously employed in the quenching problem, it will be shown how analytical solutions to the two problems just presented may be obtained with nominal computational effort. The formulations presented in the two ensuing sections present no new information. They are given for the record and to provide completeness. The final section specializes the problem to the one considered in Ref. 3 so that one may compare numerical results obtained by this method to the numerical scheme.

Radial Burning

Since the function $f_2(\psi)$ appearing in (14) must be periodic in 2π, it may be written in terms of a Fourier expansion as follows:

$$\theta_g = f_1(t)\left[\theta_{\text{avg}} + \sum_{n=1}^{\infty} A_n \cos n\psi\right] \tag{15}$$

where

$$\theta_{\text{avg}} = \frac{1}{2\pi}\int_0^{2\pi} f_2(\psi)d\psi$$

The differential equation governing heat conduction in the wall of the shell, in cylindrical coordinates, is

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\psi^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\theta}{\psi\tau} \tag{16}$$

and since. in this case, there is no z dependence, one obtains

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\psi^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial\tau}$$

The most general linear boundary conditions for the heat conduction problems are the following: at the interior face of the shell, $r = r_i$,

$$-K(\partial\theta/\partial r) = h_1(\theta_g - \theta) \qquad t > 0 \tag{17}$$

and at the exterior face, $r = r_e$,

$$-K(\partial\theta/\partial r) = h_2(\theta - \theta_0) \qquad t > 0 \tag{18}$$

where h_1 and h_2 are the boundary conductances at the interior and exterior faces, respectively, and θ_0 is the exterior ambient. The initial condition is

$$\theta(r, \psi, 0) = g(r, \psi) \tag{19}$$

In Ref. 3 the approximation is introduced that h_2 and $g(r, \psi)$ are zero. For a discussion relating to the validity of this assumption, the reader is referred to that report.

Modifying (18) and (19) in conformity with the foregoing and introducing the following dimensionless variables: $T = \theta/\theta_{\text{avg}}$, $\tau = \alpha t/r_i^2$, $R = r/r_i$, and $H_1 = h_1 r_i/K$, the boundary value problem becomes

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{1}{R^2}\frac{\partial^2 T}{\partial\psi^2} = \frac{\partial T}{\partial\tau} \tag{20}$$

$$-\partial T/\partial R = H_1(T_g - T) \qquad R = 1, t > 0 \tag{21}$$

$$-\partial T/\partial R = 0 \qquad R = R_e = r_e/r_i, t > 0 \tag{22}$$

$$T(R, \psi, 0) = 0 \qquad 1 \leqslant R \leqslant R_e \tag{23}$$

where

$$T_g = \theta_g/\theta_{\text{avg}} = f_1(\tau)\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right] \tag{24}$$

Applying the Laplace transform to Eqs. (20–22 and 24) yields

$$\frac{\partial^2\bar{T}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{T}}{\partial R} + \frac{1}{R^2}\frac{\partial^2\bar{T}}{\partial\psi^2} = s\bar{T} \tag{25}$$

$$-\partial\bar{T}/\partial R = H_1(\bar{T}_g - \bar{T}) = 0 \qquad R = 1 \tag{26}$$

$$-\partial\bar{T}/\partial R = 0 \qquad R = R_e \tag{27}$$

and

$$\bar{T}_g = \bar{f}_1(s)\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right] \tag{28}$$

Noting that (25) together with the boundary conditions are separable, the solution is assumed to be of the form

$$\bar{T}(r, \psi; s) = \xi(R; s)\sigma(\psi; s) \tag{29}$$

Substituting (28) and (29) in (26) and (27) gives

$$-(\xi' + H_1\xi)\eta = H_1\bar{f}_1(s)\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right] \qquad R = 1 \tag{30}$$

and

$$-\xi'\eta = 0 \qquad R = R_e \tag{31}$$

Hence by choosing

$$\eta = s\bar{f}_1(s)\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right]$$

the satisfaction of the boundary conditions merely requires that

$$-(\xi' + H_1\xi) = H_1/s \qquad R = 1 \tag{32}$$

$$\xi' = 0 \qquad R = R_e \tag{33}$$

From (29) and (25), one obtain

$$\left[\xi'' + \frac{1}{R}\xi'\right]\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right] - \frac{\xi}{R^2}\sum_{n=1}^{\infty} n^2 a_n \cos n\psi = s\xi\left[1 + \sum_{n=1}^{\infty} a_n \cos n\psi\right] \tag{34}$$

The foregoing implies that there must be a set of ξ's corresponding to the integers $n = 0, 1, 2 \ldots$, each of which satisfies the boundary conditions (32) and (33) together with the following differential equation:

$$R^2\xi_n'' + R\xi_n' - (n^2 + sR^2)\xi = 0 \qquad n = 0, 1, 2 \ldots \tag{35}$$

This completes the formulation of the problem for the case of nonsymmetric radial burning.

End Burning

Referring to Fig. 7, the assumption is made that, since the temperature of the burned products is considerably higher than that of the unburned fuel, a first approximation to the temperature in the chamber is the step function (Fig. 8):

$$\theta_g = \begin{cases} \theta_b & \zeta \geqslant z \geqslant 0 \\ 0 & \zeta < z \end{cases} \tag{36}$$

where $\zeta = Ut$, U is the burning rate (length per unit time), and θ_b is the burning temperature of the particular solid fuel.

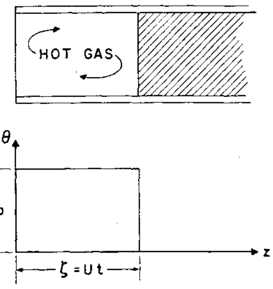


Fig. 8 End burning

In this case there is axial variation of the temperature but no angular variation. The additional assumption now is made that there is no heat flow out of or into the ends of the casing, i.e.,

$$-K(\partial\theta/\partial z) = 0 \qquad z = 0, z = l \quad (37)$$

Introducing the same nondimensional variables as in the radial burning case, with the exception that $T = \theta/\theta_b$ and $Z = z/r_i$, the boundary value problem for this case is

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{\partial^2 T}{\partial Z^2} = \frac{\partial T}{\partial \tau} \quad (38)$$

$$-\partial T/\partial R = H_1(T_g - T) \qquad R = 1 \quad (39)$$

$$\partial T/\partial R = 0 \qquad R = R_e \quad (40)$$

$$\partial T/\partial Z = 0 \qquad Z = 0, L(L = l/r_i) \quad (41)$$

and

$$T(R, Z, 0) = 0 \quad (42)$$

Here T_g is the step function:

$$T_g = \begin{cases} 1 & 0 \leqslant Z \leqslant (Ur_i/\alpha)\tau \\ 0 & Z > (Ur_i/\alpha)\tau \end{cases} \quad (43)$$

To effect a solution, it is most expedient to expand the step function (43) in a Fourier series of the form

$$\sum_{m=0}^{\infty} B_m \cos\frac{m\pi Z}{L}$$

the coefficients being functions of time. The cosine series is chosen in order to satisfy (41) a priori. Performing the necessary integrations, one obtains

$$T_g = V\tau + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\cos\frac{m\pi Z}{L}\sin m\pi V\tau \quad (44)$$

where $V = Ur_i/\alpha L$.

Taking the Laplace transform of (38–40 and 44), there results

$$\frac{\partial^2 \bar{T}}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{T}}{\partial R} + \frac{\partial^2 \bar{T}}{\partial Z^2} = s\bar{T} \quad (45)$$

$$-\partial\bar{T}/\partial R = H_1(\bar{T}_g - \bar{T}) \qquad R = 1 \quad (46)$$

$$\partial\bar{T}/\partial R = 0 \qquad R = R_e \quad (47)$$

and

$$\bar{T}_g = \frac{V}{s^2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{m\pi V}{s^2 + m^2\pi^2V^2}\cdot\frac{1}{m}\cos\frac{m\pi Z}{L} \quad (48)$$

Again assuming that $\bar{T}(R, Z; s)$ is separable, i.e.,

$$\bar{T}(R, Z; s) = \Gamma(R; s)\omega(Z; s) \quad (49)$$

the boundary conditions are satisfied if

$$\omega(Z; s) = \frac{V}{s} + s\sum_{m=1}^{\infty}\frac{m\pi V}{s^2 + m^2\pi^2V^2}\cdot\frac{1}{m}\cos\frac{m\pi Z}{L} \quad (50)$$

and

$$\begin{aligned} -(\Gamma' + H_1\Gamma) &= H_1/s \qquad R = 1 \\ \Gamma' &= 0 \qquad R = R_e \end{aligned} \quad (51)$$

From (45), following the line of reasoning employed previously, one obtains finally the differential equation to be satisfied by Γ and subject to the conditions (51):

$$R\Gamma_m'' + \Gamma_m' - (m^2\pi^2 + s)R\Gamma_m = 0 \quad (52)$$
$$m = 0, 1, 2 \ldots.$$

Solution of Eqs. (35) and (52)

Since the same method of approach is to be employed for each of the problems, the two can be discussed simultaneously. Let the operator L represent that defined in either (35) or (52), and let the dependent variable be $\sigma(R)$. The solution is sought, therefore, to

$$L(\sigma) = 0 \quad (53)$$

subject to

$$\begin{aligned} -\sigma' + H_1\sigma &= H_1/s \qquad R = 1 \\ \sigma' &= 0 \qquad R = R_e \end{aligned} \quad (54)$$

Following the procedure employed previously in the Quenching problem, let

$$\sigma = \lambda + \varphi \quad (55)$$

where λ is any function satisfying, a priori, the given boundary conditions (54), and φ is a function to be determined satisfying homogeneous boundary conditions. Substituting (55) in (53) and (54) gives

$$L(\varphi) = -L(\lambda) \quad (56)$$

$$\begin{aligned} -\varphi' + H_1\varphi &= 0 \qquad R = 1 \\ \varphi' &= 0 \qquad R = R_e \end{aligned} \quad (57)$$

and

$$\begin{aligned} -\lambda' + H_1\lambda &= H_1/s \qquad R = 1 \\ \lambda' &= 0 \qquad R = R_e \end{aligned} \quad (58)$$

The simplest function satisfying (58), which happens to be the asymptotic solution to the problem, is $1/s$. However, to demonstrate the efficacy of the method for the more general problem, a λ is sought of the form

$$\lambda = bR + cR^2 \quad (59)$$

where b and c are to be determined so that λ satisfies (58). Substituting (59) into (58) and solving for b and c yields

$$\lambda = \beta[2 - (R/R_e)](R/R_e) \quad (60)$$

where

$$\beta = -R_e^2H_1/s/2[R_e(1 - H) - 1 + H/2]$$

The solution to (56) and (57) for the given λ is sought, assuming

$$\varphi = \sum_{K=0}^{N}\bar{C}_kP_k(R) \quad (61)^{\|}$$

where $P_k(R)$ are polynomials in R, each of which satisfies (57). Clearly, the polynomial of lowest degree which can be made to satisfy (57) and still allow nontrivial boundary values is a quadratic. Hence

$$P_0 = 1 + \alpha_1R + \alpha_2R^2 \quad (62)$$

‖ The coefficients are functions of the transform parameter s. Hence they are shown barred.

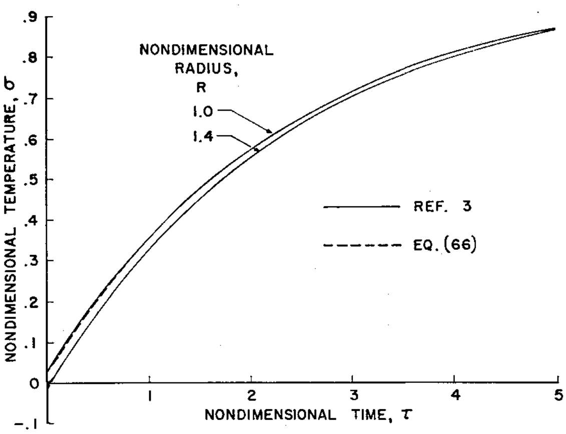


Fig. 9 $H_1 = 0.2, R_e = 1.4$

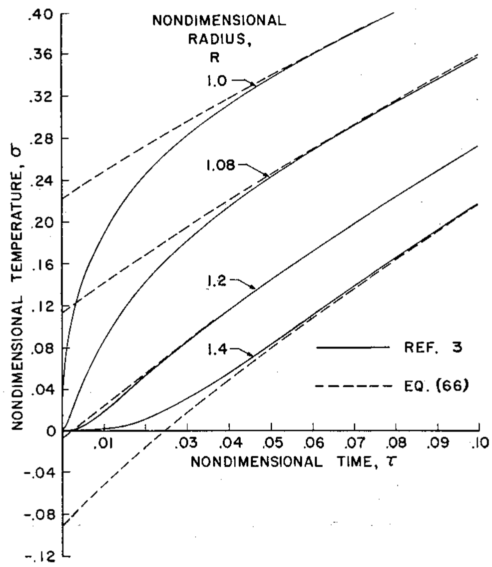


Fig. 10 $H_1 = 2.0, R_e = 1.4$

From (57) and (62), one therefore obtains

$$P_0 = 1 - \beta[2(R/R_e) - (R^2/R_e^2)] \tag{63}$$

where β is as previously defined. The second polynomial P_1 is obtained by choosing a cubic

$$P_1 = R + \alpha_3 R^2 + \alpha_4 R^3$$

and solving for α_3 and α_4 in a similar manner. To complete the set, one simply adds

$$P_n = (R-1)^n (R-R_e)^2 \qquad n = 2, 3 \ldots$$

or

$$P_n = (R-1)^2 (R-R_e)^n \qquad n = 2, 3 \ldots$$

Confining one's attention to a one-term approximation, the evaluation of $\bar{C}_0^{(1)}$ is obtained from the algebraic equation:

$$\int_1^{Re} L(\varphi^{(1)})P_0 dR = -\int_1^{Re} L(\lambda)P_0 dR \tag{64}$$

where $\varphi^{(1)} = \bar{C}_0^{(1)} P_0$.

Inasmuch as numerical results are to be compared to those given in Ref. 3, take $n = m = 0$ in (35) and (52), the resulting differential equation in either case reducing to

$$L(\varphi) = R\varphi'' + \varphi' - sR\varphi = 0 \tag{65}$$

From (65) and (64) the coefficient is

$$\bar{C}_0^{(1)} = -(1/s) + [Q/(u - sv)] \tag{66}$$

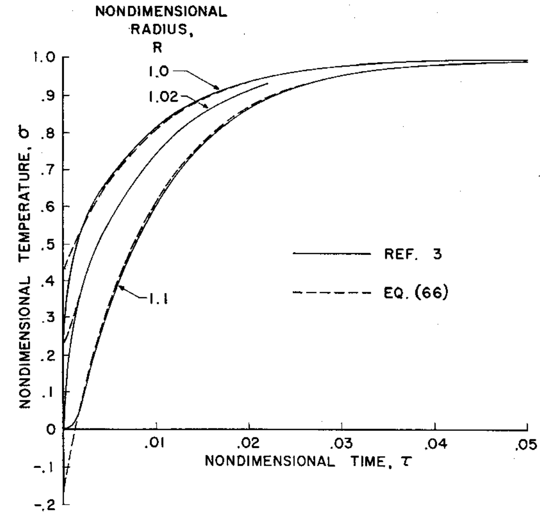


Fig. 11 $H_1 = 20.0, R_e = 1.1$

where

$$Q = -\frac{(R_e^2 - 1)}{2} + \frac{2\beta}{3R_e}(R_e^3 - 1) - \frac{\beta}{4R_e^2}(R_e^4 - 1)$$

$$u = -\frac{2\beta}{Re}(R_e - 1) + \frac{2\beta(1+\beta)}{R_e^2}(R_e^2 - 1) - \frac{10\beta^2}{3R_e^3}(R_e^3 - 1) + \frac{\beta^2}{R_e^4}(R_e^4 - 1)$$

$$v = \frac{R_e^2 - 1}{2} - \frac{4\beta}{3R_e}(R_e^3 - 1) + \frac{\beta}{2}\frac{(1+2\beta)}{R_e^2}(R_e^4 - 1) - \frac{4\beta^2}{5R_e^3}(R_e^5 - 1) + \frac{\beta^2}{6R_e^4}(R_e^6 - 1)$$

And finally, taking the inverse transform to $\bar{C}_0^{(1)}$, the approximate solution to the problem is

$$\begin{aligned}\sigma^{(1)} &= \lambda + \varphi^{(1)} \\ &= 1 - (Q/v)e^{(u/v)\tau}\{1 - \beta[(2R/R_e) - (R^2/R_e^2]\}\end{aligned} \tag{67}$$

Since u/v is a negative number, it is seen that (67) has the proper behavior as $\tau \to \infty (\sigma^{(1)} \to 1)$.

Numerical results have been obtained for the following parameters: $H_1 = 0.2$, $R_e = 1.4$; $H_1 = 2\,0$, $R_e = 1.4$; $H_1 = 20$, $R_e = 1.1$; these are plotted in Figs. 9–11, along with the values obtained by finite differences in Ref. 3. Again, as in the quenching problem, it is seen that, except for the initial instants, there is close agreement between the solutions obtained by the two methods.

Conclusions

It has been shown that analytical expressions may be obtained for the transient temperature for the quenching of an elliptical domain and the heating of a cylindrical shell. These solutions are, in general, valid for times different from the initial instant. For very early times, convergence is poor. This limitation, however, is not a consequence of the Galerkin method; rather it is a property of all solutions of the heat equation when expressed as a sum of its eigenfunctions.

When comparing the forementioned results to the Kirkpatrick solution, the term "exact" solution was not referred to, since both his solution and the analysis given here are equally exact.# The value of the Galerkin method, as

This is so because the convergence of the Galerkin procedure has been established previously. The difference in the two methods is analogous to the expansion of a function in terms of a Taylor series as compared to its Fourier expansion.

indicated in the cases considered herein, is that a single term in the Galerkin procedure yields almost the complete solution, whereas the solutions obtained by the classical methods, when the latter can be applied, converge much more slowly, thus requiring a number of terms before a significant part of the solution is obtained.

With regard to the domain containing holes, any number or shape of holes may be included provided each shape is expressible in terms of a single analytical curve, $f(x, y) = 0$, which does not generate zero values elsewhere in the domain. For example, for a rectangular hole bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$, the function $(x - a)(x - b)(y - c)(y - d) = 0$ certainly defines the hole, but unfortunately it generates zero values away from the hole, so that this function may not be used as part of the solution.** Obviously the external boundary of the domain, whether or not the domain is multiply connected, must not be re-entrant for the same reason.

** An approximation not having this deficiency is

$$\left[x - \frac{[(b+a)/2]}{(b-a)/2}\right]^{2n} + \left[y - \frac{[(d+c)/2]}{(d-c)/2}\right]^{2n} - 1 = 0$$

which, if n is sufficiently large, resembles a rectangle with rounded corners.

In the case of the cylindrical shell, the solution presented in (67) gives good results as to both behavior in time and variation in thickness. To improve the accuracy merely requires additional terms in the assumed form of solution (61). Finally, the inclusion of terms to account for angular or axial temperature variation, or both, as given by (35) and (52) presents no difficulty. It is clear that in this case $\bar{C}_0^{(1)}$ would include m or n terms. The solution corresponding to (67) would be, therefore, an infinite series over m or n, the general coefficient of which is easily determined.

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